RESIDUAL STRESSES IN SURFACE-HARDENED OIL FIELD PUMP RODS

R. E. HANSLIP, The National Supply Company, Toledo, O.

ABSTRACT

A detailed account of the method of determining the residual stresses in a solid round rod by removing concentric layers of metal from inside and outside surfaces based on the method originally set forth by George Sachs is given. SR-4 gages are used to measure the strains. The method of cementing them to the surface of a bore is explained. An example of numerical substitution into Sachs' equations, corrections to be applied to the stresses obtained, and how the stresses determined are checked, is included. The longitudinal residual stresses of a flame-hardened, an induction-hardened and a gas-carburized rod are shown and discussed. The Appendix contains a development of the Sachs' equations for the determination of residual stresses when concentric layers of material are removed from the outside surface of a cylinder.

NOMENCLATURE

E = Modulus of Elasticity - lbs. per sq. in.
F = Area of circle whose diameter equals that of the outside of the rod.
F_0 = Area of bore.
Λ = Total Longitudinal Strain = $\lambda + \mu\upsilon$.
λ = Longitudinal Strain.
Θ = Total Tangential Strain = $\upsilon + \mu\lambda$.
υ = Tangential Strain.
σ_1 = Longitudinal Stress.
σ_t = Tangential Stress.
σ_r = Radial Stress.
μ = Poisson's Ratio.

INTRODUCTION

The oil well drilling industry uses reciprocating pumps for moving fluid under high pressure. This fluid is a mixture of water, clay, abrasives and sometimes corrosive agents. Induction-hardened piston rods have failed by spalling in the area contacted by the packing with a frequency that demanded the temporary discontinuance of their manufacture. Such rods have an average case depth of 0.120 in., a hardness of Rc 60, and are made of AISI-A-4140. Gas-carburized rods do not spall nearly as often, but the manufacturing process is not as attractive. Figure 1 shows a typical failure by spalling of an induction-hardened rod.

Engineers and consultants concerned with this spalling problem, concluded that this type of failure, which was a comparative rarity in carburized rods, was a function of the residual stresses induced by the hardening process. This opinion led to the decision to institute a program to determine the residual stresses in piston rods hardened and heat treated by various methods. Accordingly, some fifty samples of piston rods were machined away in determining the residual stresses originally present in them. The method of determining the residual stresses is that devised by George Sachs (1)*.

STRESS DETERMINATION BY BORING

The determination of the residual stresses in a cylindrical specimen by removing concentric layers of material by boring has been

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* Numbers in parentheses refer to similarly numbered references in bibliography at end of paper.

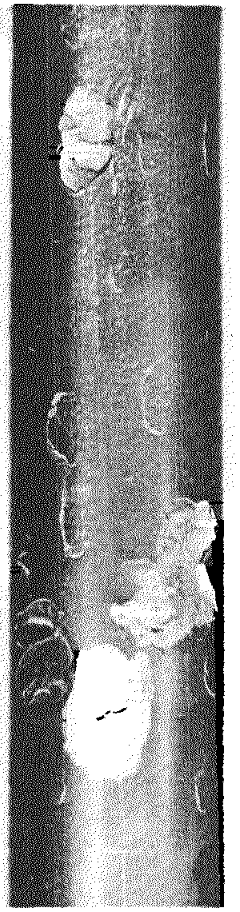

FIGURE 1. TYPICAL SPALLED PISTON ROD.

performed many times by many investigators (2, 3). Strain measurements have been made much simpler and more accurate with the advent of the fine-wire electric resistance strain gage into the field of stress analysis. This author has little to add to the general knowledge of residual stress determination by boring, but a technique developed for mounting the SR-4 strain gages was found very useful and is described in a succeeding paragraph.

In the beginning of the program the rod specimens were bored and ground internally until only a shell of approximately 0.030 wall-thickness remained. The remaining tube was then split longitudinally to release the remaining tangential stresses. Changes in strain were measured with four pairs of type A-1, SR-4 gages arranged in longitudinal and circumferential directions. Figure 2 shows two specimens just prior to splitting, one before and one after the removal of the Petrosene wax. Enameled #28 copper wire has been used for leads which are connected to the Baldwin-Southwark strain indicator directly.

On later specimens the number of gages was reduced to two pairs of gages and the method of spacing and orienting them is as shown in Figure 3. A-11 type gages were used. The paper containing the strain sensitive wires of the A-11 gage is passed through a slit in thin drawing paper and cemented in place. When the drawing paper is wrapped around a rod the strain sensitive portion of the gage is against the rod surface as it is intended that it should be. The red felt and the end of the gage containing the lead wires remains on the opposite side of the paper. While this assembly is drying it is wrapped on a rod to preform it. Mounting the gages quickly and accurately, now becomes an easy matter.

After the internal boring and grinding is completed, the residual stresses in the remaining wall must be found by extrapolating a curve or by splitting the tube and cutting tongues containing the longitudinal gages. Both methods have been followed and quite often gave widely differing stresses at the outside surface. This was unfortunate as the stresses at the surface were of more importance than the stresses at the center. As a consequence of these still unknown stresses at the surface, a program of external metal removal was started.

STRESS DETERMINATION BY REDUCING THE OUTSIDE DIAMETER

Getting started on the internal removal was a long and arduous task, but many new difficulties were encountered in external metal removal. To begin with, the equations and definition of terms found in several sources were not in agreement. The concensus of the several sources could hardly be taken with any assurance that it was correct. Consequently, the Sachs' equations for determining residual stresses by removing concentric layers of material externally had to be derived, and this derivation is to be found in the Appendix to this paper.

Measuring of strains must be done from a

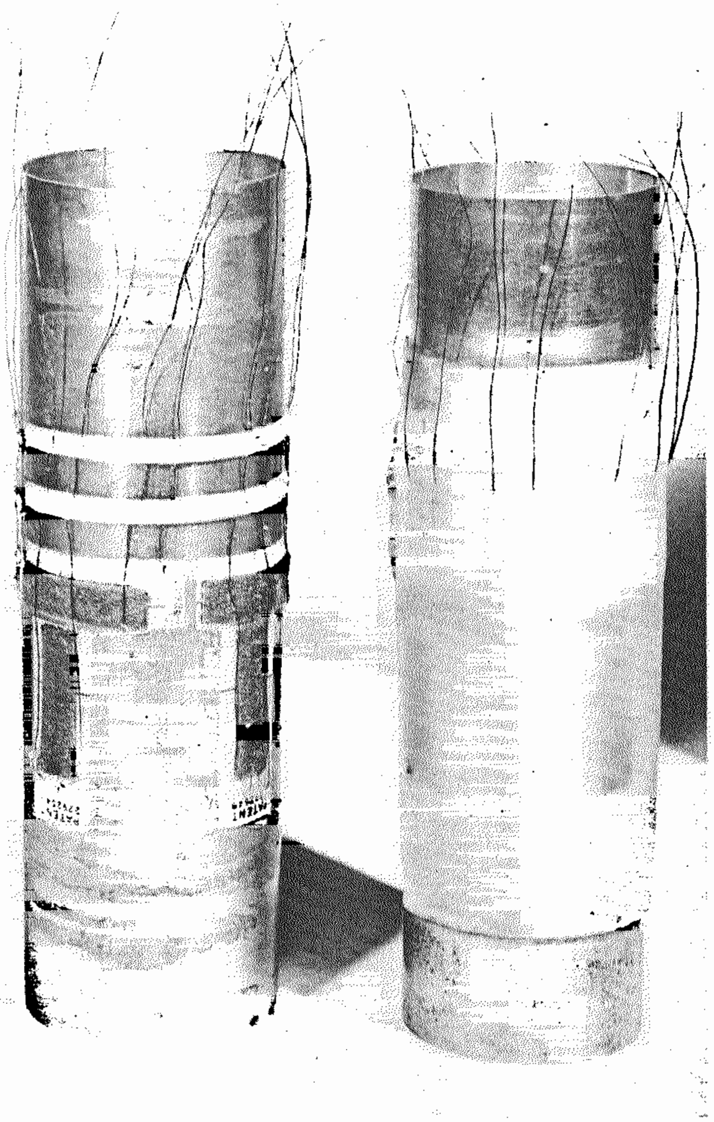

FIGURE 2. BORED-OUT SPECIMEN.

bore put into the specimen especially for this purpose. Inside the bore must be mounted the gages to measure the longitudinal and tangential strains. This sounds like a difficult task at first but with careful planning and preparation it can be done successfully. Four A-11 gages were mounted simultaneously in a 1.262 diameter bore by the method of mounting the gages on paper first. Naturally the copper leads were soldered on and taped in place before the assembly was cemented in the bore. After placing the assembly in the bore with a generous amount of cement, the gage assembly was pressed against the sides of the bore with the aid of an inflated penny balloon. The gages are insulated with Petrosene wax. The uniform layer of wax was obtained by pouring the wax in the bore around a hollow tube. Running hot water through the tube enabled the tube to be extracted easily.

During the grinding and turning operations on the outside surface, the specimen was mounted on a mandrel. The gage lead wires were coiled inside the bore. Coiled lead

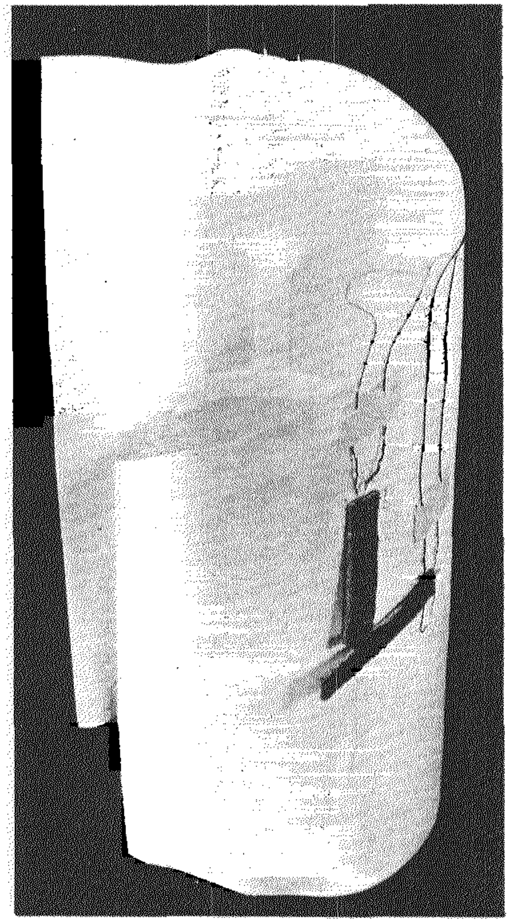

FIGURE 3. METHOD OF MOUNTING A-11 GAGES.

wires and the mandrel can be seen in Figure 4. The eight #23 copper wire leads were coiled and uncoiled twenty-five times in a 1.262 bore without breaking due to bending fatigue.

In grinding, metal was removed at the rate of one-thousandth inch on the diameter per pass of the grinding wheel. Specimens were cooled during grinding by copious quantities of coolant. After each operation consisting of the machining away of one-tenth of a square inch of metal or more, the specimens were stored alongside the dummy gage specimen for approximately 18 hours before taking the strain reading, as only one layer of metal was removed per day. After the hardened case had been entirely ground away, the remaining material was turned off until the wall thickness became so thin that grinding had to be resumed. The final operation consisted of tongueing and splitting the tube to release the longitudinal and tangential stresses.

When a hole is introduced into a specimen containing residual stresses, the residual stresses are changed from what existed in the original specimen. Consequently, corrections must be made to the stresses obtained from gages mounted in a bore if the stresses in the original specimen are to be known. To illustrate the corrections and to furnish an example

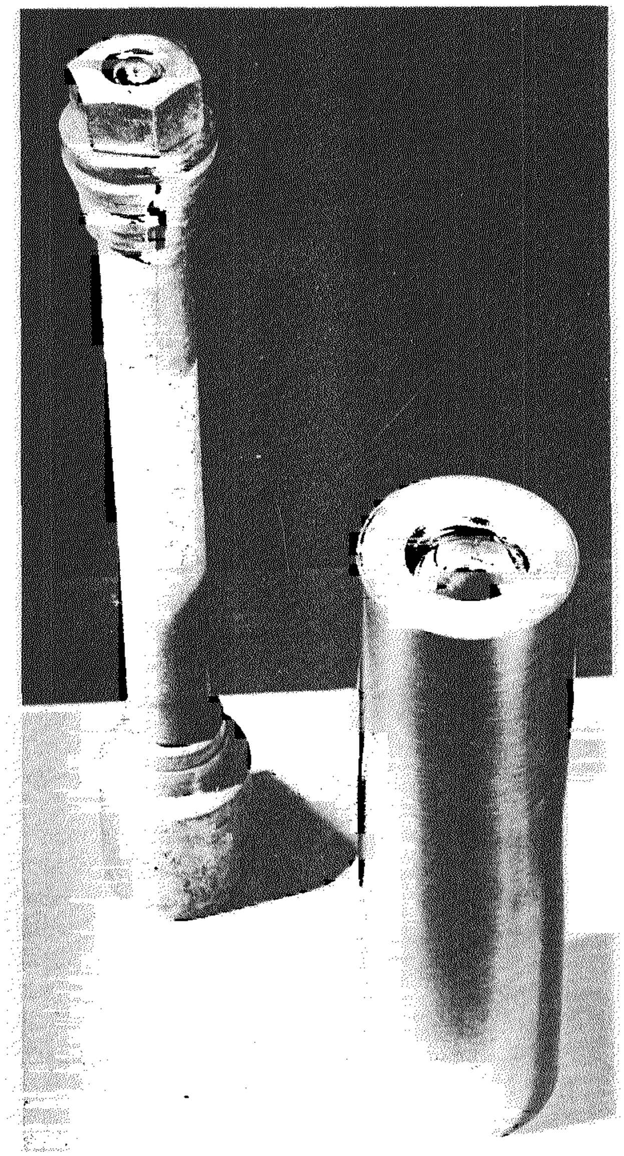

FIGURE 4. SPECIMEN WITH GAGES MOUNTED IN BORE.

of a numerical substitution in the Sachs' equations, the program on a specific rod specimen will be followed.

STRESS DETERMINATION BY REMOVAL OF METAL FROM BOTH INSIDE AND OUTSIDE SURFACES

A flame-hardened specimen of approximately AISI-5160 composition 2.625 inches in diameter and 8 inches long was bored out to an inside diameter of 1.262 inches in five steps of one-quarter square inch of area per step. Four A-11, SR-4 strain gages had been mounted on the outside to measure the longitudinal and tangential strains. From the data obtained in this way, the residual stresses in the center of the specimen were obtained by the use of the Sachs' boring-out equations. These are plotted in Figure 5.

In the bore, four more A-11 gages were mounted and insulated as described above. The specimen was ground and turned in twenty-five steps until a wall thickness of only 0.025 inches remained. The first seventeen operations removed about one-tenth square inch of area per operation. The final operation was the tongueing and splitting of the remaining shell to release the last remaining stresses. Figure 6 shows the final state of this specimen and also the installation of gages.

For an example of numerical substitution to find the residual stresses from Sachs' equations, an area F, of 4.5 sq. in., was arbitrarily

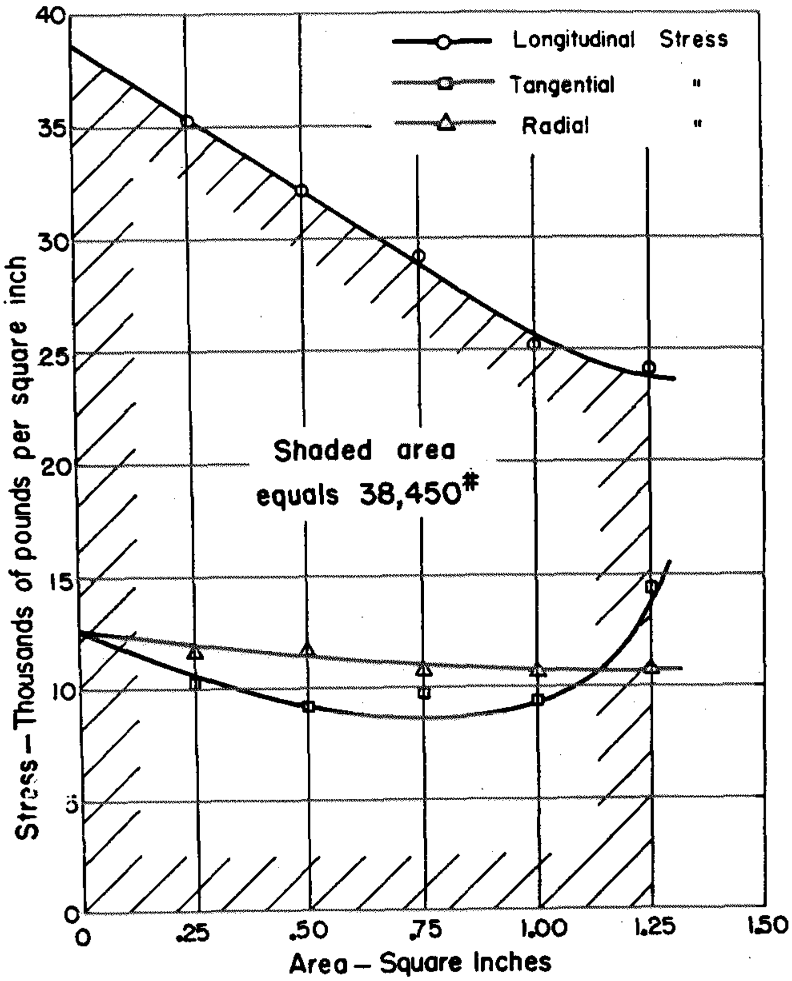

FIGURE 5. RESIDUAL STRESSES IN BORE.

selected. This corresponds to a radius of 1.197 in. The longitudinal and tangential strains, λ and ν, as recorded at this area are -438 and -356 microinches, respectively. Λ and Θ are calculated as follows:

$$\Lambda = \lambda + \mu\nu = -438 + .3(-356) = -545 \text{ microinches.}$$

$$\Theta = \nu + \mu\lambda = -356 + .3(-438) = -487 \text{ microinches.}$$

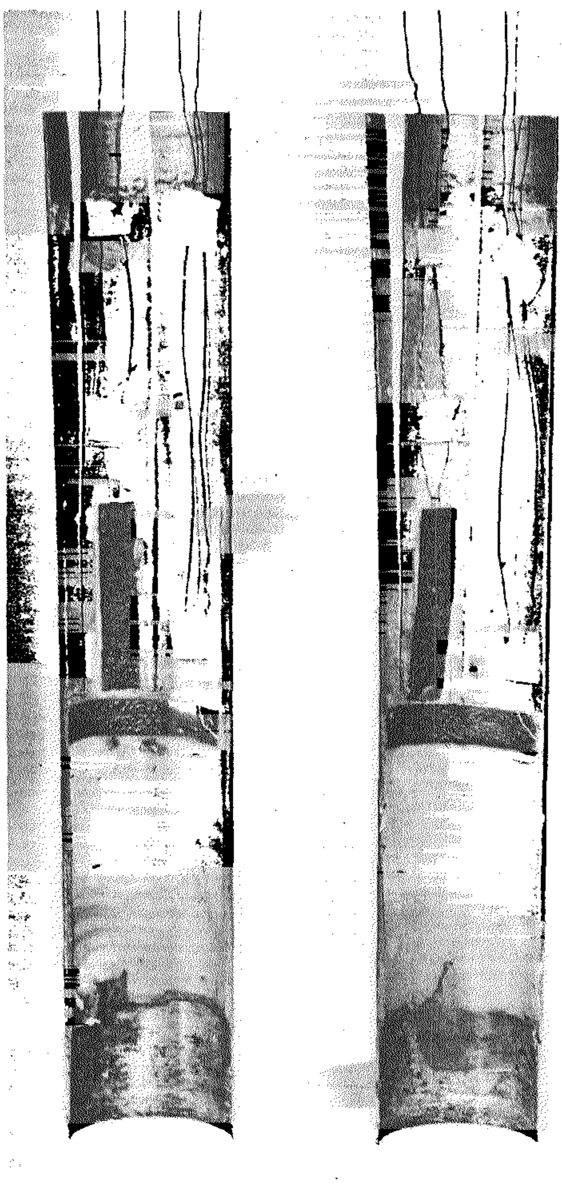

FIGURE 6. SPLIT AND TONGUED SPECIMEN.

Values of Λ and Θ obtained after each step in the removal of material are plotted in Figure 7, and a curve is faired through each set of points. It looks like the abscissa is backwards but this is done to make the slopes $d\Lambda/dF$ and $d\Theta/dF$ come out correctly when the slope is measured in the usual manner. This method of combining the strain data and plotting the curves is different from the method employed by most experimenters, but it yielded excellent results here (4, 5).

The values of Λ and Θ as determined from the curves when F equals 4.5, are -560 and -485 microinches, respectively. The slopes $d\Lambda/dF$ and $d\Theta/dF$ of the curves where F equals 4.5 in. are -617 and -535 respectively. From this information the residual stresses are calculated from the equations developed in the Appendix to this paper.

$$\frac{E}{1-\mu^2} = \frac{30 \times 10^6}{1-(.3)^2} = 32.97 \times 10^6 \text{ lbs. per sq. in.}$$

$$F - F_0 = 4.5 - 1.25 = 3.25 \text{ sq. in.}$$

$$\sigma_l = 32.97 \times 10^6 \left[\frac{3.25(-617)-(-560)}{10^6}\right] = -47{,}460 \text{ lbs. per sq. in.}$$

$$\frac{F + F_0}{2F} = \frac{4.5 + 1.25}{2 \times 4.5} = .639$$

$$\sigma_t = 32.97 \times 10^6 \left[\frac{3.25(-535) - .639(-485)}{10^6}\right] = -47{,}080 \text{ lbs. per sq. in.}$$

$$\sigma_r = -32.97 \times 10^6 \left[\frac{\frac{3.25}{2 \times 4.5}(-485)}{10^6}\right] = 5{,}770 \text{ lbs. per sq. in.}$$

As stated above, these stresses are the stresses in the tube and not those in the original solid specimen. The residual stresses in the core are used to obtain the corrections. From Figure 5 it is seen that the longitudinal force removed is 38,450 lbs. This force divided by the remaining tube's area is the stress relieved by removing the core and it must be prefixed with a negative sign and added to the longitudinal stresses found in the remaining tube. If this force were to be applied by some external means to the tube it would result in a compressive stress of 9,250 lbs. per sq. in. in the tube.

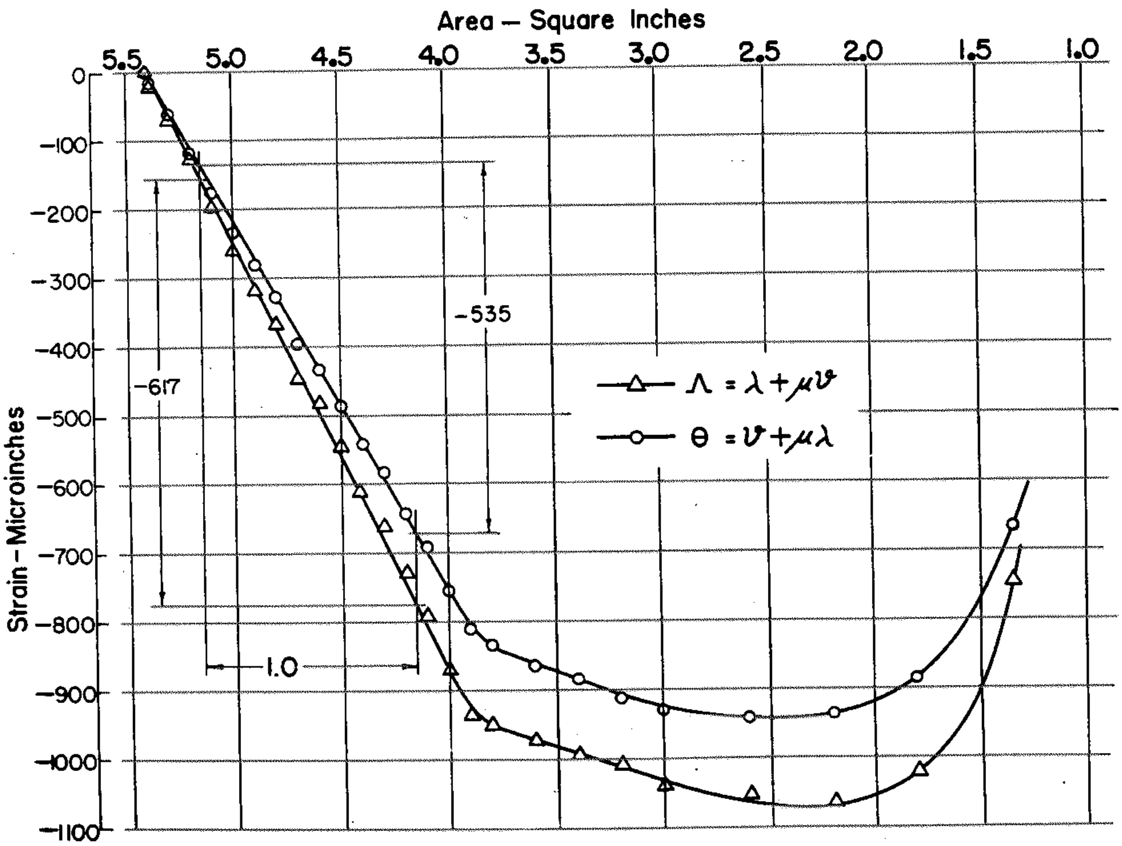

FIGURE 7. TOTAL STRAIN CURVES.

$$\frac{38{,}450}{5.407-1.25} = 9{,}250 \text{ lbs. per sq. in.}$$

$$-47{,}640 - 9{,}250 = -56{,}890 \text{ lbs. per sq. in.}$$

For the corrections to the tangential and radial stresses the formulas for thick-walled cylinders are used. When the center was bored out to 1.262 in., radial tensile stresses were released at the bore surface. This is equivalent to introducing a pressure into a thick-walled cylinder of the dimensions of the remaining tube. The stresses induced at a radius r in the wall of the tube are given by the following equations (6):

$$\sigma_t = \frac{a^2 p_1}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) \text{ and}$$

$$\sigma_r = \frac{a^2 p_1}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right)$$

where a is the inside radius and b is the outside. p_1 equals the radial stress, which is obtained from Figure 5. At a radius of 1.197 in., corresponding to an area of 4.5 sq. in., the change in stress due to the release of the tensile radial stress is

$$\sigma_t = \frac{(.631)^2(10{,}680)}{(1.312)^2-(.631)^2}\left[1 + \frac{(1.312)^2}{(1.197)^2}\right] = 7{,}070 \text{ lbs. per sq. in.}$$

$$\sigma_r = \frac{(.631)^2(10{,}680)}{(1.312)^2-(.631)^2}\left[1 - \frac{(1.312)^2}{(1.197)^2}\right] = -650 \text{ lbs. per sq. in.}$$

These stresses must also be changed in sign and added to the tangential and radial stresses found by grinding down the remaining tube. The sign change is necessary because tensile radial forces in the bore would induce compressive tangential and tensile radial stresses in the tube.

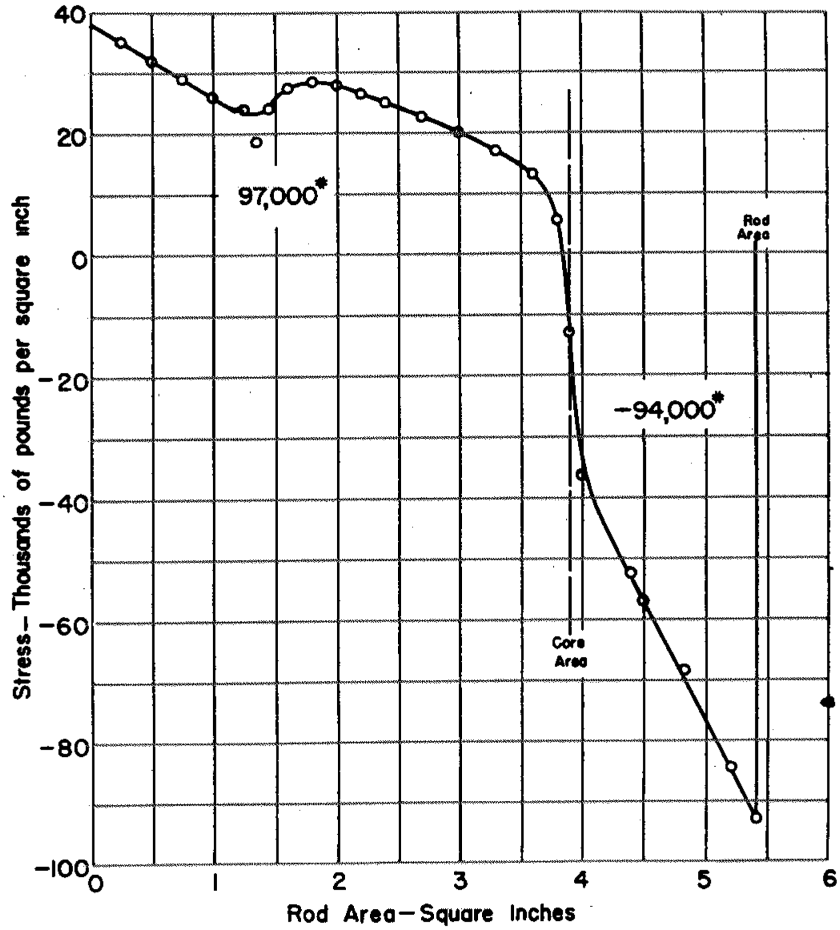

FIGURE 8. LONGITUDINAL STRESSES IN A FLAME-HARDENED ROD.

$\sigma_t = -47{,}080 + (-7{,}070) = -54{,}150$ lbs. per sq. in.

$\sigma_r = 5{,}770 + 650 = 6{,}420$ lbs. per sq. in.

These stresses are those which existed in the solid rod and they are shown in Figures 8, 9 and 10, together with the stresses found to be present in the bore area. It will be noticed that a few experimentally determined points do not lie on the stress curves in the area of the bore and the final outside diameter. This is probably due to the unreliability of the dA/dF and $d\Theta/dF$ values obtained from the total strain curves of both the internal and external metal removal phases. The slope of the curves where the strain data ends is questionable since the curves were made to pass through the last data point. Stresses calculated from the data obtained by splitting and tongueing the remaining shell are not presented because of their nonconformity with the curve. The reason for this disparity is not known, but it might be for the same reason that a completely bored-out specimen gives an indication of stress reversal in the outside skin when split. Both phenomena could be attributed to compressive stresses induced in the bore by the machining process.

CHECKING STRESSES

When the stresses at each point of the cross-section of the original solid rod have been found, they may be subjected to an equilibrium check. The longitudinal forces on a transverse

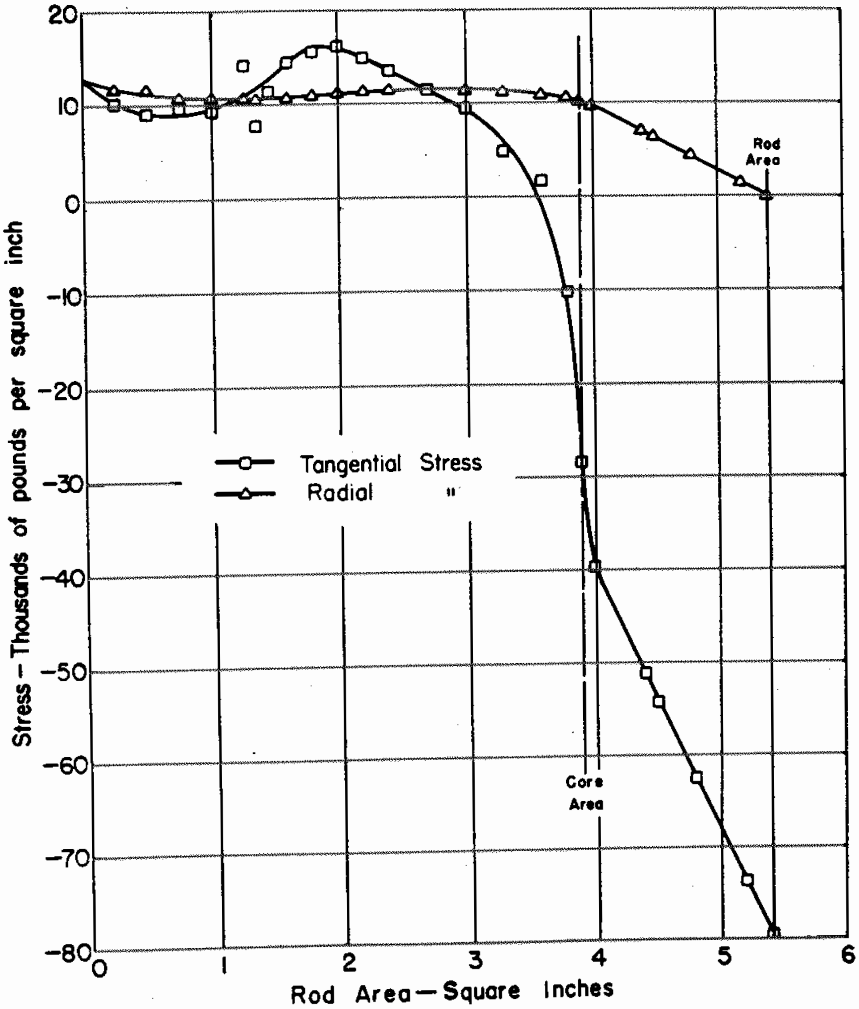

FIGURE 9. TANGENTIAL AND RADIAL STRESSES IN A FLAME-HARDENED ROD.

cross-section must equal zero if a condition of equilibrium is to exist. When the longitudinal stresses are plotted against transverse rod area, with zero area corresponding to the rod centerline as in Figure 8, the integrated areas between the stress curve and the line of zero stress represent forces. These positive and negative forces should equal zero. In Figure 8 the force values of the integrated areas have been recorded thereon. If the shape of the curve is taken to be correct, the origin need be shifted only a few hundred pounds per square inch to make their sum equal to zero.

The sum of the forces in a diametrical section must also equal zero for the solid rod to be in equilibrium. If the tangential stresses are plotted against the rod radius, the integrated area between the stress curve and the line of zero stress represents a force per inch of rod length. It is this force which should equal zero. In Figure 10 the force per inch values of the positive and negative areas are recorded therein. The line of zero stress has to be shifted less than 50 lbs. per sq. in. to make the areas equal.

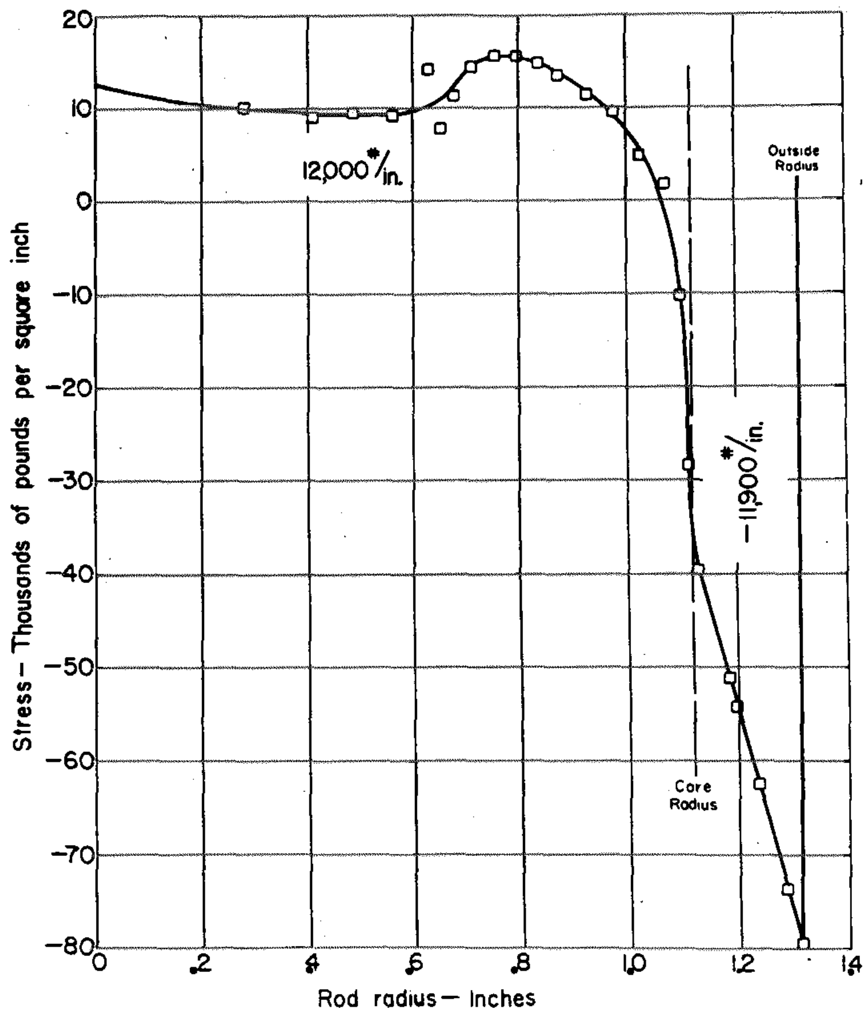

FIGURE 10. TANGENTIAL STRESSES IN A FLAME-HARDENED ROD.

RESULTS

In almost all of the rods subjected to a residual stress investigation, it was found that the longitudinal and tangential stress curves were generally of the same shape. The magnitudes of the peak stresses on the tangential curve were always less than those on the longitudinal stress curve. For this reason the longitudinal stress curve is used for comparison purposes. It is generally acknowledged that the radial stresses are of little importance (4, 5), so they are not considered either.

Figures 11 and 12 are the longitudinal stress curves of an induction-hardened and a gas-carburized rod, respectively. They were 2.25 inches in diameter. The gas-carburized specimen was manufactured of AISI-A-4620 steel. The stresses of these specimens were determined in a slightly different manner from that described above. Two eight-inch portions of the same rod were subjected to the two different methods of metal removal: internal and external. The stress curves obtained were corrected as above and matched together in their overlapping regions.

The longitudinal stress curves of the three piston rods are considerably different from each other. The maximum compressive stress of the flame-hardened rod is on the surface of the rod, whereas the other two have their maximum beneath the surface. The stress gradient in the induction-hardened specimen is much steeper than in the gas-carburized specimen and the magnitude of the peak stresses is much greater also. Rate of change of stress at the surface of the rod has been taken to be significant in predicting the spalling of piston rods.

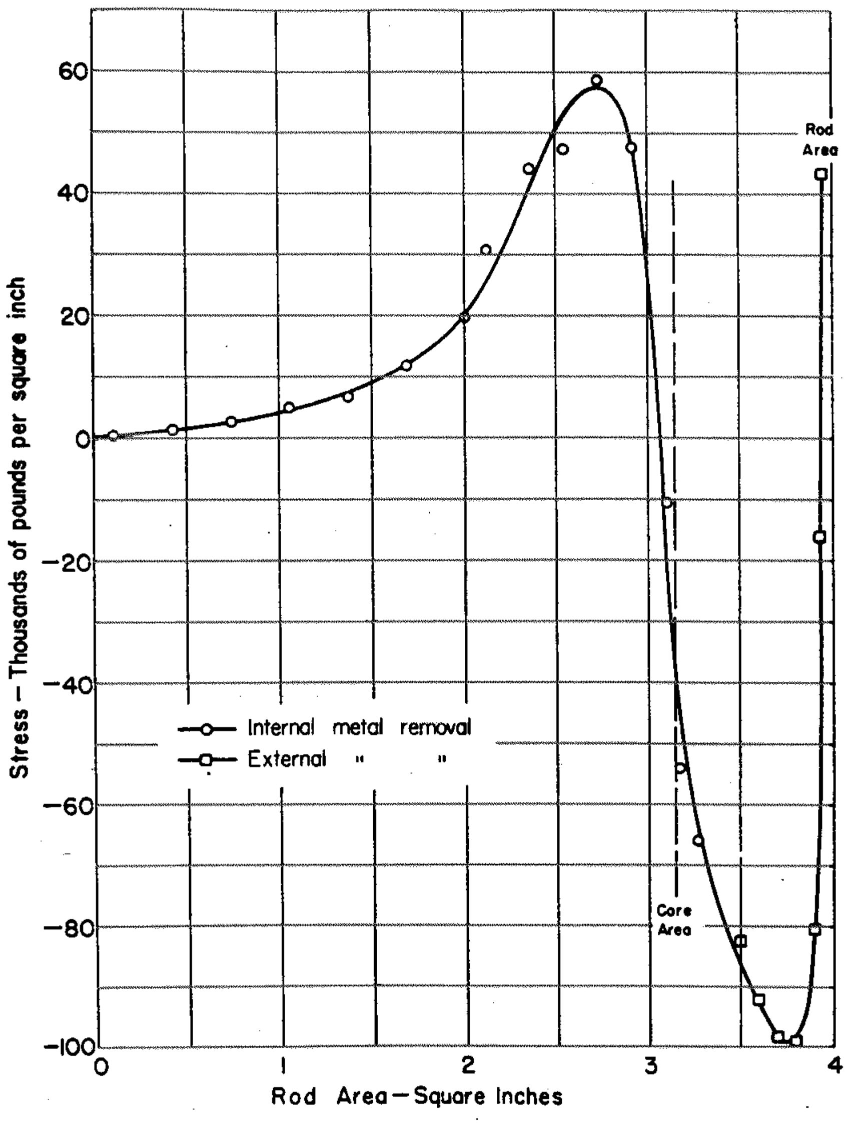

FIGURE 11. LONGITUDINAL STRESSES IN AN INDUCTION-HARDENED ROD.

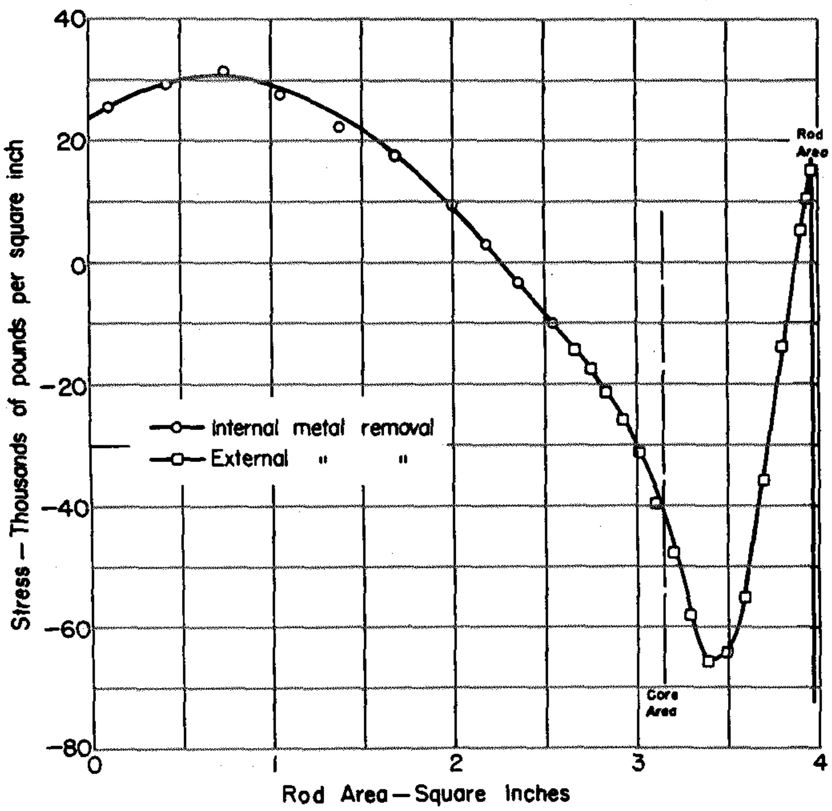

FIGURE 12. LONGITUDINAL STRESSES IN A GAS-CARBURIZED ROD.

APPENDIX

DERIVATION OF THE EQUATIONS FOR THE DETERMINATION OF RESIDUAL STRESSES IN TUBES BY REMOVING LAYERS OF MATERIAL FROM THE OUTSIDE OF THE TUBE

One of the assumptions made in determining residual stresses by this method is that the removal of a layer produces an equal change in longitudinal stress at all points on the cross-section. Figure 13 illustrates the change which takes place in the longitudinal stresses when a layer of thickness ΔR is removed from a tube of outside radius b and inside radius a. Curve ABCDE is the longitudinal stress distribution in the wall of the tube before removal of the layer ΔR. It has the *origin* o-o. After removal of the layer ΔR, the origin shifts to position o'-o', i.e., the value of the longitudinal stress at every point of the wall changes by the amount $d\sigma_1$, and the new stress curve is the line ABCD. Note that the above assumption requires that the shape of the stress curve in the remaining material be the same before and after removal of the layer. Note also that because the stress changes by the same value throughout the thickness of the wall, the strain changes in the longitudinal direction would be the same at each point in the wall.

Consider a tube of outside diameter $2b$ and

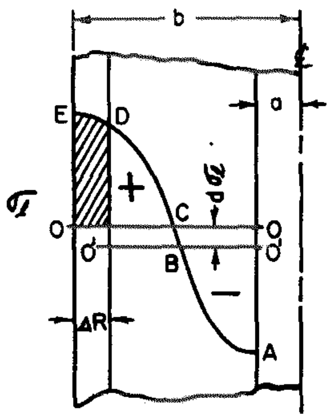

FIGURE 13.

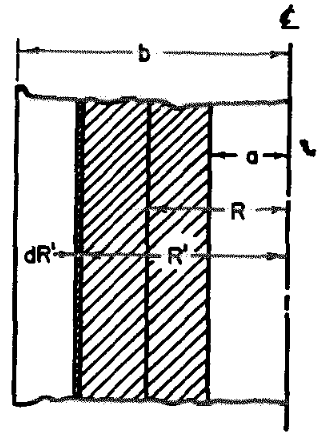

FIGURE 14.

inside diameter $2a$. Since layers of metal will be removed from the outside surface, a means of measuring the longitudinal strains, λ, and the tangential strains, υ, on the surface of the bore is necessary. When an outside diameter of $2R$ is reached, there exist stresses on the surface, σ_l'', σ_t'', and σ_r''. At radius R of the original tube existed stresses σ_l, σ_t, and σ_r. It is the stresses σ_l, σ_t, and σ_r which are to be determined. The subscripts l, t, and r, indicate longitudinal, tangential and radial directions, respectively. In the process of removing all the material outside of radius R, there have been added to the stresses σ_l, σ_t, and σ_r, magnitudes of stress σ_l', σ_t', and σ_r', which can be defined as follows:

$$\sigma_l' = \sigma_l'' - \sigma_l, \tag{1}$$

$$\sigma_t' = \sigma_t'' - \sigma_t, \tag{2}$$

$$\sigma_r' = \sigma_r'' - \sigma_r. \tag{3}$$

The above equations can also be derived in the following manner. Consider the longitudinal stress in the layer dR' of Figure 14. Assume that it is in tension. If it is removed, the remaining metal expands, indicating an increase in strain. This increase in strain can be translated into an increase in stress, $d\sigma_{l_a}'$, where the subscript "a" refers to the stress at surface "a". This stress, $d\sigma_{l_a}'$, will be of the same sign and magnitude as the change in stress in surface R, $d\sigma_l'$. $d\sigma_l'$ must be added algebraically to the stress which originally existed at surface R, σ_l. When all of the layers, dR', have been removed up to R and the total change in stress is σ_l', the stress existing at surface R is now σ_l''. From the foregoing, it is seen that σ_l, σ_l', and σ_l'' can be combined in the following equation:

$$\sigma_l + \sigma_l' = \sigma_l''. \tag{1a}$$

The same explanation can be extended to the case where a compressive stress is removed in a layer and the resulting strain is negative in sign. The corresponding stress must be added algebraically to the original stress, σ_l.

In an analogous fashion the following equations involving the tangential and radial stresses, are formed.

$$\sigma_t + \sigma_t' = \sigma_t'', \tag{2a}$$

$$\sigma_r + \sigma_r' = \sigma_r''. \tag{3a}$$

It remains to define σ' and σ'' in terms of the measured strains, λ and υ.

It was mentioned above that an increase in strain could be translated into an increase in stress. From the theory of elasticity the following relations between the stresses and strains on the inside surface of a tube are obtained:

$$\cdot \; E\lambda = \sigma_{l_a} - \mu\sigma_{t_a}, \tag{4}$$

$$E\nu = \sigma_{t_e} - \mu\sigma_{l_e}, \qquad (5)$$

where σ_{l_e} and σ_{t_e} are the longitudinal and tangential stresses respectively on the inside surface of the tube.

Solving for σ_{l_e} and σ_{t_e},

$$\sigma_{l_e} = \frac{E}{1-\mu^2}(\lambda + \mu\nu), \qquad (6)$$

$$\sigma_{t_e} = \frac{E}{1-\mu^2}(\nu + \mu\lambda). \qquad (7)$$

A small change in λ and ν, $d\lambda$ and $d\nu$ respectively, produces a small change, $d\sigma_{l_e}$ and $d\sigma_{t_e}$, in σ_{l_e} and σ_{t_e}, according to the following formulas derived from (6) and (7).

$$d\sigma_{l_e} = \frac{E}{1-\mu^2}(d\lambda + \mu d\nu), \qquad (8)$$

$$d\sigma_{t_e} = \frac{E}{1-\mu^2}(d\nu + \mu d\lambda). \qquad (9)$$

CALCULATION OF σ_l', σ_t', AND σ_r'

In the derivation of equation (1a) it was said that

$$d\sigma_{l_e}' = d\sigma_l', \qquad (10)$$

when a layer, dR', is removed from the outside. Substituting for $d\sigma_{l_e}'$, the expression for $d\sigma_{l_e}$ from (8) and integrating to find the effect of removing all the layers dR' down to surface R, the following expression for σ_l' is obtained:

$$\sigma_l' = \frac{E}{1-\mu^2}(\lambda + \mu\nu). \qquad (11)$$

To determine σ_t' and σ_r', consider the effect on the stresses σ_t and σ_r at surface R, of removing the layer dR'. This has the effect of imposing an external pressure, $d\sigma_{r_1}$, equal to the radial stress, at R'.

The change in tangential stress at any point r in the cylinder wall, due to a change in external pressure, $d\sigma_{r_1}$, is expressed by the following formula from the theory of thick-walled cylinders (6):

$$d\sigma_t' = \frac{-(R')^2}{(R')^2 - a^2}\left(1 + \frac{a^2}{r^2}\right) d\sigma_{r_1}, \qquad (12)$$

where R' is the outside radius, and a is the inside radius of the tube. Applying this formula to the point in the cylinder where r equals R, the following expression for $d\sigma_t'$ is obtained.

$$d\sigma_t' = \frac{-(R')^2}{(R')^2 - a^2}\left(1 + \frac{a^2}{R^2}\right) d\sigma_{r_1}. \qquad (13)$$

At the point on the cylinder where the change in strain, $d\lambda$ and $d\nu$, is measured, i.e., where r equals a, a change in stress $d\sigma_{t_e}$ due to $d\sigma_{r_1}$ occurs, which is also obtained by proper substitution in (12).

$$d\sigma_{t_e} = \frac{-(R')^2}{(R')^2 - a^2}\left(1 + \frac{a^2}{a^2}\right) d\sigma_{r_1}. \qquad (14)$$

By eliminating $d\sigma_{r_1}$ between (13) and (14), substituting for $d\sigma_{t_e}$ the expression for $d\sigma_{t_e}$ given by (9), solving for $d\sigma_t'$ and integrating, one obtains the following expression for σ_t':

$$\sigma_t' = \frac{E}{1-\mu^2}\left(\frac{F + F_a}{2F}\right)\theta, \qquad (15)$$

where F equals the area of a circle of radius R, F_a equals the area of a circle of radius a, or the area of the bore of the cylinder, and $\theta = \nu + \mu\lambda$.

The change in radial stress $d\sigma_r'$, is found in a similar manner from the equation for the change in radial stress at any point r in a thick-walled cylinder when subjected to an external pressure $d\sigma_{r_1}$.

$$d\sigma_r' = \frac{-(R')^2}{(R')^2 - a^2}\left(1 - \frac{a^2}{r^2}\right) d\sigma_{r_1}. \qquad (16)$$

For this case, r again equals R, and the formula becomes:

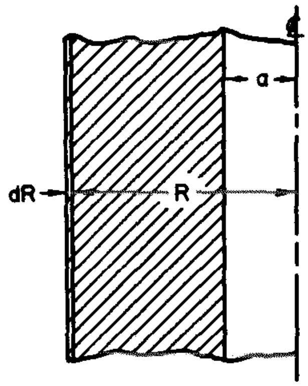

FIGURE 15.

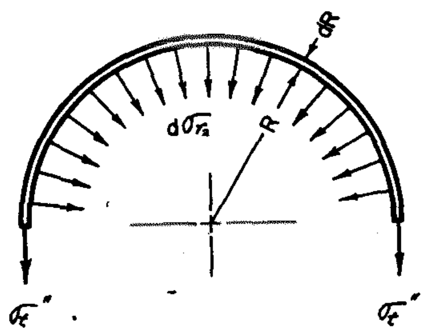

FIGURE 16.

$$d\sigma_r' = \frac{-(R')^2}{(R')^2 - a^2}\left(1 - \frac{a^2}{R^2}\right) d\sigma_{r_a}. \qquad (17)$$

By eliminating $d\sigma_{r_a}$ simultaneously from (17) and (14), substituting for $d\sigma_{t_a}$ the expression for $d\sigma_{t_a}$ from (9), solving for $d\sigma_r'$ and integrating, one obtains the following expression for σ_r' :

$$\sigma_r' = \frac{E}{1-\mu^2}\left(\frac{F - F_a}{2F}\right)\theta, \qquad (18)$$

where the terms are defined as previously.

CALCULATION OF σ_l^*, σ_t^*, AND σ_r^*

To determine the longitudinal stress remaining on the outside surface of the cylinder after removing all the material down to a radius R, consider the equilibrium of longitudinal forces between the thin layer, dR, to be removed, and the change in longitudinal stress that will take place in the remaining material after the layer is removed. As has been said before, the change in longitudinal stress at any point in the remaining metal is equal to the change in stress at any other point when a layer of material is removed from the outside. Therefore, the change in longitudinal stress can be represented by the change in stress at radius a, which is $d\sigma_{l_a}$. The stress in the layer dR to be removea is σ_l^*. Equate the longitudinal forces in the layer dR and the remaining material. Referring to Figure 15,

$$2\pi R dR \sigma_l^* = \pi(R^2 - a^2) d\sigma_{l_a}. \qquad (19)$$

Substituting for $d\sigma_{l_a}$ the expression for $d\sigma_{l_a}$ from (8) and solving for σ_l^*, one obtains the following expression for σ_l^* :

$$\sigma_l^* = \frac{E}{1-\mu^2}\left(\frac{F - F_a}{dF}\right)(d\lambda + \mu d\upsilon), \qquad (20)$$

or

$$\sigma_l^* = \frac{E}{1-\mu^2}(F - F_a)\frac{d\Lambda}{dF}, \qquad (21)$$

because $F = \pi R^2$ and $dF = 2R\pi dR$.

$\Lambda = \lambda + \mu\upsilon$ and $d\Lambda = d\lambda + \mu d\upsilon$.

The value of σ_t^* as a function of the measured strains may be determined by finding the effect on the tangential stresses at radius a, of removing the thin layer, dR, containing the stress σ_t^*. Such a shell of radius R and thickness dR, was before removal subjected to a tangential stress, σ_t^*, and an internal pressure, which is the same as the radial stress, $d\sigma_{r_a}$, at the inside surface (See Figure 16). The equation expressing the equilibrium of forces on this half-shell is:

$$2\sigma_t^* dR + 2R d\sigma_{r_a} = 0. \qquad (22)$$

The effect on the measured strains of removing the thin layer dR, is due to the removal of the radial stress, $d\sigma_{r_2}$. This may be considered to be equivalent to the imposition of a pressure $d\sigma_{r_2}$ on the external surface of a thick-walled cylinder. In such a case, the change in tangential stress, $d\sigma_{t_a}$, at radius a, is given by (12) where r equals a and $d\sigma_t'$ therefore becomes $d\sigma_{t_a}$, R' equals R, and $d\sigma_{r_1}$, is now $d\sigma_{r_2}$.

$$d\sigma_{t_a} = \frac{-R^2}{R^2 - a^2} \times 2 \times d\sigma_{r_2}. \tag{23}$$

Substituting the expression for $d\sigma_{t_a}$ from (9) and the value of $d\sigma_{r_2}$ in terms of σ_t'', R, and dR, from (22), in (23), one obtains the following expression for σ_t'':

$$\sigma_t'' = \frac{E}{1-\mu^2}(F - F_a)\frac{d\Theta}{dF}, \tag{24}$$

because

$\Theta = \nu + \mu\lambda$ and $d\Theta = d\nu + \mu d\lambda$ and $dF = 2\pi R dR$.

σ_r'' is the radial stress on the external surface of the cylinder after it has been reduced to radius R. It must necessarily equal zero as any radial stress must, on a free surface.

Substituting in equations (1), (2) and (3) the expressions derived for σ_l', σ_l'', σ_t', σ_t'' and σ_r', and solving for σ_l, σ_t, and σ_r, one obtains the desired equations:

$$\sigma_l = \frac{E}{1-\mu^2}\left[(F - F_a)\frac{d\Lambda}{dF} - \Lambda\right], \tag{25}$$

$$\sigma_t = \frac{E}{1-\mu^2}\left[(F - F_a)\frac{d\Theta}{dF} - \left(\frac{F + F_a}{2F}\right)\Theta\right], \tag{26}$$

$$\sigma_r = -\frac{E}{1-\mu^2}\left[\frac{F - F_a}{2F}\right]\Theta, \tag{27}$$

where σ_l, σ_t, and σ_r are the longitudinal, tangential, and radial stresses at the radius corresponding to area F.

μ = Poisson's Ratio.

Λ = $\lambda + \mu\nu$

λ = Strain in Longitudinal Direction.

ν = Strain in Tangential Direction.

Θ = $\nu + \mu\lambda$

F = Cross-sectional Area of the Cylinder after removal of each layer, neglecting the area of the bore. (Equal to πR^2).

F_a = Area of bore = πa^2

$F - F_a$ = Remaining cross-sectional area of cylinder.

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